$(f \circ g)(x) = f(g(x))$ Due Sun

$\vec{x} \to T_A(\vec{x}) \to T_B(T_A(\vec{x})) = (T_B \circ T_A)(\vec{x})$

## 1.9 – Compositions of Matrix Transformations

The **composition of $T_B$ with $T_A$** is achieved by first applying the matrix transformation $T_A$ to a vector and then applying the matrix transformation $T_B$ to the image vector. We denote the composition of $T_B$ with $T_A$ by $T_B \circ T_A$ which is read "$T_B$ circle $T_A$." This is also expressed as $(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x}))$.

**Theorem 1.9.1** If $T_A: R^n \to R^k$ and $T_B: R^k \to R^m$ are matrix transformations, then $T_B \circ T_A$ is also a matrix transformation, and $T_B \circ T_A = T_{BA}$.

Pf: Let $T_A(\vec{x}) = A\vec{x}$ and $T_B(\vec{x}) = B\vec{x}$ be as described.

$$(T_B \circ T_A)(\vec{x}) \overset{\text{Def of comp.}}{=} T_B(T_A(\vec{x}))$$

$T_A$ is a matrix transf.

$$= T_B(A\vec{x})$$

$$= B(A\vec{x})$$

$T_B$ is a matrix transf.

$$= (BA)\vec{x}$$

assoc. prop. of matrix mult.

8. Find the standard matrix for the stated composition in $R^2$.

a. A rotation about the origin of $60°$, followed by an orthogonal projection onto the $x$-axis, followed by a reflection about the line $y = x$.

b. An orthogonal projection onto the $x$-axis, followed by a rotation about the origin of $45°$, followed by a reflection about the $y$-axis.

c. A rotation about the origin of $15°$, followed by a rotation about the origin of $105°$, followed by a rotation about the origin of $60°$.

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Rightarrow R_{60^\circ} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$T_A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad T_B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$T_B \circ T_A \circ R_{60^\circ} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}}$$

$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 \\ -2x_1 + x_2 \\ -x_1 - 3x_2 \end{bmatrix}$$

formula

12. Let $T_1(x_1, x_2, x_3) = (4x_1, -2x_1 + x_2, -x_1 - 3x_2)$ and $T_2(x_1, x_2, x_3) = (x_1 + 2x_2, -x_3, 4x_1 - x_3)$.

a. Find the standard matrices for $T_1$ and $T_2$.

b. Find the standard matrices for $T_2 \circ T_1$ and $T_1 \circ T_2$.

c. Use the matrices obtained in part (b) to find formulas for $T_1(T_2(x_1, x_2, x_3))$ and $T_2(T_1(x_1, x_2, x_3))$.

matrix

a. $[T_1] = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix}$, $[T_2] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix}$

b. $[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 3 & 0 \\ 17 & 3 & 0 \end{bmatrix}$

$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 4 & 8 & 0 \\ -2 & -4 & 1 \\ -1 & -2 & 3 \end{bmatrix}$

c. $T_2(T_1(x_1, x_2, x_3)) = (2x_2, x_1 + 3x_2, 17x_1 + 3x_2)$

$T_1(T_2(x_1, x_2, x_3)) = (4x_1 + 8x_2, -2x_1 - 4x_2 + x_3, -x_1 - 2x_2 + 3x_3)$

If $T_A: R^n \to R^n$ is a matrix operator whose standard matrix $A$ is invertible, then $T_A$ is **invertible**, and the **inverse** of $T_A$ is $T_A^{-1} = T_{A^{-1}}$.

20. Determine whether the matrix operator $T: R^3 \to R^3$ defined by the equations is invertible; if so, find the standard matrix for the inverse operator, and find $T^{-1}(w_1, w_2, w_3)$.

a.

$w_1 = x_1 - 2x_2 + 2x_3$
$w_2 = 2x_1 + x_2 + x_3$
$w_3 = x_1 + x_2$

$[T_A] = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Invertible? $\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

We find $\boxed{[T_A]^{-1} = \begin{bmatrix} 1 & -2 & 4 \\ -1 & 2 & -3 \\ -1 & 3 & -5 \end{bmatrix}}$ $\boxed{\text{So } T \text{ is invertible}}$

Work:

$$\begin{array}{ccc|ccc} & 4 & 3 & & & \\ 1 & \boxed{-2} & \boxed{2} & 1 & 0 & 0 \\ \boxed{2} & 1 & \boxed{1} & 0 & 1 & 0 \\ \boxed{1} & \boxed{1} & 0 & 0 & 0 & 1 \\ 1 & 2 & & & & \end{array}$$

$R_2 \Rightarrow R_2 - 2R_1$

$$\begin{array}{cccccc} 2 & 1 & 1 & 0 & 1 & 0 \\ -2 & 4 & -4 & -2 & 0 & 0 \\ \hline 0 & 5 & -3 & -2 & 1 & 0 \end{array}$$

$R_3 \Rightarrow R_3 - R_1$

$$\begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 2 & -2 & -1 & 0 & 0 \\ \hline 0 & 3 & -2 & -1 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 5 & -3 & -2 & 1 & 0 \\ 0 & 3 & -2 & -1 & 0 & 1 \end{array}\right]$$

$R_3 \Rightarrow 3R_2 - 5R_3$

$$\begin{array}{cccccc} 0 & 15 & -9 & -6 & 3 & 0 \\ 0 & -15 & 10 & 5 & 0 & -5 \\ \hline 0 & 0 & 1 & -1 & 3 & -5 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 5 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 3 & -5 \end{array}\right]$$

$R_1 \Rightarrow R_1 - 2R_3$

$$\begin{array}{cccccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 2 & -6 & 10 \\ \hline 1 & -2 & 0 & 3 & -6 & 10 \end{array}$$

$R_2 \Rightarrow R_2 + 3R_3$

$$\begin{array}{cccccc} 0 & 5 & -3 & -2 & 1 & 0 \\ 0 & 0 & 3 & -3 & 9 & -15 \\ \hline 0 & 5 & 0 & -5 & 10 & -15 \end{array}$$

Then $R_2 \Rightarrow \frac{1}{5} R_2$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 3 & -6 & 10 \\ 0 & 1 & 0 & -1 & 2 & -3 \\ 0 & 0 & 1 & -1 & 3 & -5 \end{array}\right]$$

$R_1 \Rightarrow R_1 + 2R_2$

$$\begin{array}{cccccc} 1 & -2 & 0 & 3 & -6 & 10 \\ 0 & 2 & 0 & -2 & 4 & -6 \\ \hline 1 & 0 & 0 & 1 & -2 & 4 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 4 \\ 0 & 1 & 0 & -1 & 2 & -3 \\ 0 & 0 & 1 & -1 & 3 & -5 \end{array}\right]$$

$$\boxed{T^{-1}(w_1, w_2, w_3) = (w_1 - 2w_2 + 4w_3, -w_1 + 2w_2 - 3w_3, -w_1 + 3w_2 - 5w_3)}$$

b.

$$w_1 = x_1 - 3x_2 + 4x_3$$
$$w_2 = -x_1 + x_2 + x_3$$
$$w_3 = -2x_2 + 5x_3$$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 + R_1$

$$\begin{array}{cccccc} -1 & 1 & 1 & 0 & 1 & 0 \\ 1 & -3 & 4 & 1 & 0 & 0 \\ \hline 0 & -2 & 5 & 1 & 1 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & 1 & 0 & 0 \\ 0 & -2 & 5 & 1 & 1 & 0 \\ 0 & -2 & 5 & 0 & 0 & 1 \end{array}\right]$$

we see here that this is not invertible